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How To Play Two-Player Restricted Quantum Games with 10 Cards

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Abstract We show that it is possible to play 'restricted' two-player quantum games proposed originally by Marinatto and Weber (Phys. Lett. A 272:291–303, 2000) by purely macroscopic means, in the simplest case having as the only equipment a pack of 10 cards. Our example shows also that some apparently 'genuine quantum' results, even those that emerge as a consequence of dealing with entangled states, can be obtained by suitable application of Kolmogorovian probability calculus and secondary-school mathematics, without application of the 'Hilbert space machinery'.

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1 Introduction

Although the theory of quantum games, originated in 1999 by Meyer [2] and Eisert, Wilkens, and Lewenstein [3] is only seven years old, numerous results obtained during these years [4] have shown that extending the classical theory of games to the quantum domain opens new interesting possibilities. Although Eisert and Wilkens [5] noticed that “Any quantum system which can be manipulated by two parties or more and where the utility of the moves can be reasonably quantified, may be conceived as a quantum game”, the extreme fragility of quantum systems may make playing quantum games difficult. In this respect it is interesting whether quantum games with all their ‘genuine quantum’ features could be played with the use of suitably designed macroscopic devices.

Another reason for which we were seeking for the macroscopic realization of quantum games is a question about probability calculus on which a game is based. Calculating expected payoffs yielded by various strategies is the most basic task of any rational player. In classical games Kolmogorovian probability calculus is used for this purpose while in quantum games probabilities of getting various results are calculated from the Hilbert-space description of a game and they are manifestly non-Kolmogorovian. In this respect our showing that the same numerical results can be obtained either by Hilbert-space calculations or by suitable application of ‘classical mathematics’ may be significant not only to the theory of quantum games but also in other situations in which probabilities are calculated with the use of the ‘Hilbert-space machinery’.

Van Enk [6] noticed that a single qubit is not a truly quantum system in the sense that its behavior can be mocked up by a classical hidden-variable model and on this basis he criticized Meyer’s [2] Quantum Penny-Flip game. Our macroscopic model shows that this criticism may be, at least in some cases, extended to quantum games that employ two qubits, even entangled ones.

Our idea of playing quantum games with macroscopic devices stems from the macroscopic devices proposed by one of us [7] that perfectly simulate the behavior and measurements performed on two maximally entangled spin-1/2 particles. For example, they allow to violate the Bell inequality with $2\sqrt{2}$, exactly ‘in the same way’ as it is violated in the EPR experiments. In order to play Marinatto and Weber’s ‘restricted’ version of two-player, two-strategies quantum games we shall not use the ‘full power’ of this device, but we give its more detailed description such that the principle of what we try to do is clear.

2 Macroscopic Simulations of Marinatto and Weber’s Quantum Games

2.1 The Quantum Machine

The quantum machine is a model for a spin-1/2 particle that allows also to model measurements of a spin along various directions. It consists of a point particle with negative charge q_0 contained in a 3-dimensional unit sphere S^2 [8, 9]. All points of the sphere represent states of the spin. Points on the surface of S^2 correspond to pure states: the spin-state

$$|\psi\rangle = \left(\cos\frac{\theta}{2}e^{\frac{-i\phi}{2}}, \sin\frac{\theta}{2}e^{\frac{i\phi}{2}}\right)$$

is represented by the point $v(1,\theta,\phi)$ on the surface of S^2. Interior points of the sphere S^2 represent mixed states:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & r\sin\theta e^{-i\phi} \\ r\sin\theta e^{i\phi} & 1 - r\cos\theta \end{pmatrix}$$

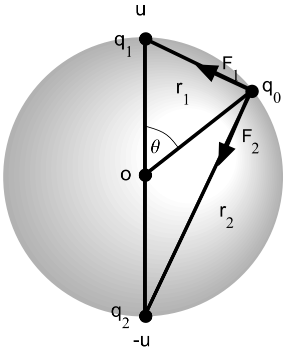

Fig. 1 The macroscopic quantum machine

is represented by $v(r,\theta,\phi)$. The point $v(0,\theta,\phi)$ in the center of the sphere represents the density matrix

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Hence states are represented equivalently as this is the case in the Bloch model for the spin 1/2.

A measurement $\alpha_{u(\theta,\phi)}$ along the direction u consists in placing a positive charge q_1 in u and a positive charge q_2 in $-u$ on the surface of the sphere. The charges q_1 and q_2 are taken at random from the interval $[0, Q]$ and their distribution within this interval is assumed to be uniform, but they have to satisfy the constraint $q_1 + q_2 = Q$. So in fact we can think that only q_1 is taken at random from the interval $[0, Q]$ and that $q_2 = Q - q_1$. If the initial state of the machine is as depicted on Fig. 1, and the measurement is said to yield an outcome ‘spin up’ (resp. ‘spin down’) when the particle is pulled to the point u (resp. $-u$), one can easily calculate [7, 8] that the obtained probability distribution of getting various results coincides with the quantum mechanical probability distribution over the set of outcomes for a spin-1/2 experiment:

$$P(\text{spin up}) = \cos^2\frac{\theta}{2}, \qquad P(\text{spin down}) = \sin^2\frac{\theta}{2}. \tag{1}$$

A macroscopic model for a quantum system of two entangled spin-1/2 particles in the singlet state [7] can be constructed by ‘coupling’ two such sphere models by adding a rigid but extendable rod with a fixed center that connects negative charges representing ‘single’ particles (Fig. 2). Because of this rod the two negative charges are ‘entangled’ since a measurement performed on one of them necessarily influences the state of the other one.

2.2 Quantum Games Proposed by Marinatto and Weber

The ‘restricted’ version of two-player, two-strategy quantum games proposed by Marinatto and Weber is as follows: The ‘quantum board’ of the game consists of two qubits that are in a definite initial state (entangled or not). Each of two players obtains one qubit and his/her strategy consists in applying to it either the identity or the spin-flip operator, or a probabilistic mixture of both. Then the state of both qubits is measured and the players get their payoffs calculated according to the specific bimatrix of the played game and the results of measurements. Marinatto and Weber in their paper [1] considered a game with a payoff bimatrix:

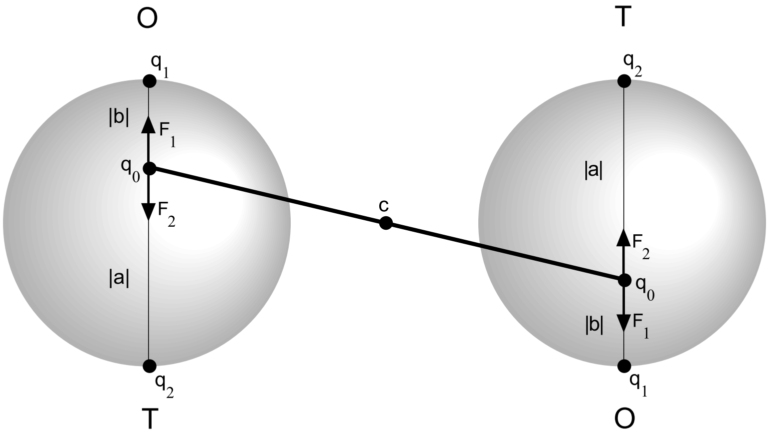

Fig. 2 The initial ‘entangled’ state of Aerts’ quantum machine

$$\begin{array}{lcc} & \text{Bob}: O & \text{Bob}: T \\ \text{Alice}: O & (\alpha, \beta) & (\gamma, \gamma) \\ \text{Alice}: T & (\gamma, \gamma) & (\beta, \alpha) \end{array} \tag{2}$$

which, if $\alpha > \beta > \gamma$, is the payoff bimatrix of the Battle of the Sexes game (Alice wants to go to the Opera while Bob prefers to watch Television, so if they both choose O Alice’s payoff $\$_A(O, O) = \alpha$ is bigger than Bob’s payoff $\$_B(O, O) = \beta$, and if they both choose T their payoffs are the opposite. Since they both prefer to stay together, if their strategies mismatch they are both unhappy and get the lowest payoff γ). Marinatto and Weber showed that if the initial state of the pair of qubits is not entangled, the quantum version of the game reproduces exactly the classical Battle of the Sexes game played with mixed strategies, but if the game begins with an entangled state of the ‘quantum board’: $|\psi_{in}\rangle = a|OO\rangle + b|TT\rangle$, $|a|^2 + |b|^2 = 1$, then the expected payoff functions for both players crucially depend on the values of squared moduli of ‘entanglement coefficients’ $|a|^2$ and $|b|^2$, and allow for new ‘solutions’ of the game not attainable in the classical or factorizable quantum case.

2.3 Marinatto and Weber’s ‘Restricted’ Quantum Game Realized by the Macroscopic Quantum Machine

Let us look now how simply Marinatto and Weber’s ‘restricted’ quantum game can be macroscopically realized with the use of the macroscopic quantum machine. We describe firstly the macroscopic realization of the game that begins with an entangled state

$$|\psi_{in}\rangle = a|OO\rangle + b|TT\rangle, \qquad |a|^2 + |b|^2 = 1. \tag{3}$$

The game that begins with a non-entangled state can be obtained from it as a limit in which either $|a|^2 = 0$ or $|b|^2 = 0$. The initial configuration of the macroscopic machine that realizes the state (3) is depicted on Fig. 2.

Applying the spin-flip operator by any of the players is realized as exchanging the labels O and T on his/her sphere. Let us note that this is a local operation since it does not influence in any way the sphere of the other player. Applying the identity operator obviously means doing nothing. When both players make (or not) their movements, the measurement is performed which, similarly to the original Aerts' proposal in [7], consists in placing a positive charge q_1 on the North pole and a positive charge q_2 on the South pole of Alice's sphere, and the same charges, respectively, on the South and North poles of Bob's sphere (i.e., on the Bob's sphere q_1 is placed on the South pole and q_2 on the North pole). Again, charges q_1 and q_2 are taken at random from the interval [0, Q] with uniform probability distribution satisfying the constraint $q_1 + q_2 = Q$. Assuming for simplicity that forces between 'left' positive and 'right' negative, resp. 'right' positive and 'left' negative charges are negligible (which can be achieved by using a rod that is long enough or by suitable screening) we can calculate the forces F_1 and F_2 between the negative charges q_0 placed at both ends of the rod and, respectively, positive charges q_1 and q_2:

$$F_1 = C\frac{q_0 q_1}{|b|^2} \quad \text{and} \quad F_2 = C\frac{q_0 q_2}{|a|^2}. \tag{4}$$

The final state of the machine (the result of measurement) depends on which force, F_1 or F_2, is bigger. If the labels O and T are placed as on Fig. 2, the result of the measurement is (O, O) iff $F_1 > F_2$, and (T, T) iff $F_1 < F_2$. The probability that $F_1 > F_2$ is as follows:

$$P(F_1 > F_2) = P(q_1|a|^2 > q_2|b|^2) = P(q_1 > Q|b|^2) \tag{5}$$

which, since q_1 is assumed to be uniformly distributed in the interval [0, Q], yields

$$P(O, O) = P(F_1 > F_2) = \frac{Q - Q|b|^2}{Q} = 1 - |b|^2 = |a|^2. \tag{6}$$

Of course in this case

$$P(T, T) = P(F_1 < F_2) = 1 - |a|^2 = |b|^2. \tag{7}$$

Let us assume, following Marinatto and Weber, that Alice applies the identity operator (in our model: undertakes no action) with probability p and applies the spin-flip operator (in our model: exchanges the labels O and T on her sphere) with probability $1 - p$, and Bob does the same on his side with respective probabilities q and $1 - q$. Consequently, when both players make (or not) their movements, the configuration depicted on Fig. 2 occurs with probability pq, and the result of the measurement is (O, O) with probability $pq|a|^2$ and (T, T) with probability $pq|b|^2$. Taking into account three other possibilities (Alice undertaking no action and Bob exchanging the labels, Alice exchanging the labels and Bob undertaking no action, and both of them exchanging their labels) which occur with respective probabilities $p(1-q)$, $(1-p)q$, and $(1-p)(1-q)$, and the payoff bimatrix (2), we obtain the following formulas for the expected payoff of Alice:

$$\begin{aligned}\overline{\$}_A(p, q) &= pq(|a|^2\alpha + |b|^2\beta) + p(1-q)\gamma \\ &\quad + (1-p)q\gamma + (1-p)(1-q)(|a|^2\beta + |b|^2\alpha) \\ &= p[q(\alpha + \beta - 2\gamma) - \alpha|b|^2 - \beta|a|^2 + \gamma] \\ &\quad + q(-\alpha|b|^2 - \beta|a|^2 + \gamma) + \alpha|b|^2 + \beta|a|^2,\end{aligned}$$

and the expected payoff of Bob:

$$\begin{aligned}\overline{\$}_B(p,q) &= pq(|b|^2\alpha+|a|^2\beta)+p(1-q)\gamma\\ &\quad+(1-p)q\gamma+(1-p)(1-q)(|a|^2\alpha+|b|^2\beta)\\ &= q[p(\alpha+\beta-2\gamma)-\alpha|a|^2-\beta|b|^2+\gamma]\\ &\quad+p(-\alpha|a|^2-\beta|b|^2+\gamma)+\alpha|a|^2+\beta|b|^2.\end{aligned}\tag{8}$$

Let us note that these formulas, although obtained from the 'mechanistic' model through 'classical' calculations based on Kolmogorovian probabilities are *exactly* the same as formulas (7.3) of Marinatto and Weber [1] for the payoff functions of Alice and Bob in their 'reduced' version of the quantum Battle of the Sexes game that begins with an entangled state (3).

The macroscopic model of the quantum game that begins with a non-entangled state $|\psi_{in}\rangle = |OO\rangle$ can be obtained by putting in (3) $a=1$ and $b=0$, which means that in this case the rod on Fig. 2 leads from the North pole of Alice's sphere to the South pole of Bob's sphere. In this case we obtain

$$\begin{aligned}\overline{\$}_A(p,q) &= p[q(\alpha+\beta-2\gamma)+\gamma-\beta]+q(\gamma-\beta)+\beta,\\ \overline{\$}_B(p,q) &= q[p(\alpha+\beta-2\gamma)+\gamma-\alpha]+p(\gamma-\alpha)+\alpha,\end{aligned}\tag{9}$$

again in perfect agreement with Marinatto and Weber's [1] formulas (3.3).

This result might be surprising since the rod connecting two particles represents entanglement in the macroscopic quantum machine so one could expect that when the initial state of the game is not entangled, this connection should be broken. However, it should be noticed that in the device depicted on Fig. 2 the rod connecting two particles is, in fact, redundant. The reason for which we left it on Fig. 2 is twofold: firstly, we wanted to stress that our idea of a macroscopic device that allows to play quantum games stems from the ideas published in [7], and secondly, this rod might be essential for macroscopic simulations of other quantum games, more general than Marinatto and Weber's 'restricted' ones.

Thus, we see that what vanishes in the 'non-entanglement' limit of the considered quantum game is the 'randomness in measurement', since now (except for the zero-probability case when $q_1=0$, $q_2=Q$) the initial state of the machine does not change in the course of the measurement whatever is the value of q_1.

2.4 Marinatto and Weber's 'Restricted' Quantum Game Realized with a Pack of 10 Cards

The lack of any importance of the connecting rod and the fact that all distances, charges, and forces in the device depicted on Fig. 2 are symmetric with respect to the middle of the rod allow to produce a still more simple model of the considered game, in fact so simple that it can be played with a piece of paper and a pack of 10 cards bearing numbers $0, 1, \dots, 9$. The game is played in three steps. In the first step the initial 'quantum' state of the game (3) is fixed. Since only the squared moduli of entanglement coefficients $|a|^2$ and $|b|^2$ are important and $|a|^2+|b|^2=1$, it is enough to fix a point representing $|a|^2$ in the interval $[0,1]$ (Fig. 3).

In the next step the players exchange, or not, labels O and T on their sides modeling in this way application of spin-flip, resp. identity, operators. In the third step a measurement is made, which is executed by choosing at random a number in the interval $[0,1]$. If a chosen number is smaller than $|a|^2$ which, if the probability distribution is uniform in $[0,1]$, happens with the probability $|a|^2$, the result of the measurement is given by labels placed by both players close to 1, otherwise by labels placed close to 0. Although random choosing

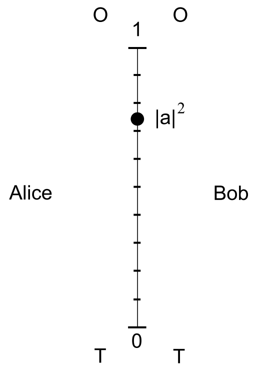

Fig. 3 The board to play 'restricted' quantum games with 10 cards

of a number may be executed in many ways, we propose to use a pack of 10 cards bearing numbers $0, 1, \ldots, 9$ which allows to draw one by one, with uniform probability, consecutive decimal digits of a number until we are sure that the emerging number is either definitely bigger or definitely smaller than $|a|^2$ (we put aside the problem of drawing in this way the number that *exactly* equals $|a|^2$ since its probability is 0, as well as the fact that in a series of n drawings we actually choose one of 10^n numbers represented by separate points uniformly distributed in the interval $[0, 1 - 10^{-n}]$). Of course calculations of the payoff functions that we made while describing the device depicted on Fig. 2 are still valid in this case, so we again obtain a perfect macroscopic simulation of Marinatto and Weber's 'restricted' two-players, two-strategy quantum games.

3 Conclusions

We have shown that one does not have to be equipped with sophisticated and costly devices and perform subtle manipulations on highly fragile single quantum objects in order to play quantum games, at least in the 'restricted' Marinatto and Weber's version: all that suffices is a piece of paper and a pack of 10 cards. Moreover, the 'random element' of the game, which in microscopic realization is a consequence of the randomness of quantum measurements is, in the proposed macroscopic realization, a consequence of random choosing a number from the interval $[0, 1]$. All calculations concerning our model are performed with the use of secondary-school mathematics which shows that 'Hilbert space machinery' is not the only tool to get 'genuine quantum' results.

It should be mentioned that another proposal of macroscopic simulations of quantum games was recently published in [10] (see also references to previous publications by the same authors cited therein). However, neither non-locality, nor entanglement which are the 'most basic' quantum phenomena and which are responsible for the biggest differences between quantum and classical games are macroscopically imitated in this paper.

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